Total Irregularity Strength of the Total Graph of a Star Graph

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ABSTRACT

An edge (A vertex) irregular total k-labeling of a graph G is such a labeling of the vertices and edges with integers 1, 2,..., k that the weights of any two different edges (vertices) are distinct, where the weight of an edge (a vertex) is the sum of the label of the edge (vertex) itself and the label of its incident vertices (edges). The minimum k, for which the graph G has an edge (a vertex) irregular total k-labeling, is called the total edge (vertex) irregularity strength of the graph G and is denoted by tes(G) (tvs(G)). In this paper, the upper bound of the total edge (vertex) irregularity strength of the total graph of the star graph $K_{1,n}$, $n \geq 2$.

INTRODUCTION

We consider only finite, simple and undirected graphs, without loops and multiple edges. Let G = (V, E) be a graph with the vertex set V and the edge set E. The total graph T(G) of a graph G is the graph with vertex set V∪E and two vertices of T(G) are adjacent whenever they are neighbors in G. A star graph S_n is a complete bipartite graph $K_{1,n}$.

An edge irregular total k-labeling of a graph G is such a labeling of the vertices and edges with integers 1, 2,..., k that the weights of any two different edges are distinct, where the weight of an edge is the sum of the label of the edge itself and the labels of the two end vertices. The minimum k for which the graph G has an edge irregular total k-labeling, is called the total edge irregularity strength of the graph G and in denoted by tes(G). Similarly, a vertex irregular k-labeling of a graph G is such a labeling of the vertices and edges with integers 1, 2,..., k that the weights of any two different vertices are distinct, where the weight of a vertex is the sum of the label of the vertex itself and the labels of its incident edges. The minimum k, for which the graph G has a vertex irregular total k-labeling, is called the total vertex irregularity strength of G and is denoted by tvs(G). The notions of the total edge irregularity strength and the total vertex irregularity strength were first introduced by Bača. et al [1] in a recent paper. They are invariants analogous to irregularity strength of a graph G [3, 4, 6, 8, 9, 11, 12]. In [1], Bača et al. put forward the lower bounds of tes(G) and tvs(G) in terms of the maximum degree Δ, minimum degree δ, the order $|E(G)|$ and the size $|V(G)|$, which may be stated as in Theorem 1.1 and 1.2.

Theorem 1.1: $tes(G) \geq \max\left\{\left\lceil \frac{\Delta+1}{2} \right\rceil, \left\lceil \frac{|E|+2}{3} \right\rceil\right\}$

Theorem 1.2: $tvs(G) \geq \left\lceil \frac{|V(G)|+\delta}{\Delta+1} \right\rceil$

Based on these theorems, Bača et al [1] determined the exact values of the total edge irregularity strength of path P_n, star S_n, wheel W_n, and friendship graph F_n and obtained the exact values of the total vertex irregularity strength of star S_n, complete graph K_n, cycle C_n and Prism D_n. Ivančo and Jendrol′ [5] determined the total edge irregularity strength of any tree. Jendrol′ et al [7] proved the exact values of the total edge irregularity strength of complete graphs and complete bipartite graphs. Misškuf and Jendrol′ [10] determined the exact values of the total edge irregularity strength of mxn grids. Brandt et al [2] proved a conjecture about edge irregular total labeling. Tong Chunling et al[13] obtained the exact values of the total edge irregularity strength of some families of graphs including the generalized Petersen graph, Ladder, Mobius band etc.

In this paper, the upper bound of the total edge (vertex) irregularity strength of the total graph of the star graph $K_{1,n}$, $n \geq 2$.

2 MAIN RESULTS

The upper bound of the total edge (vertex) irregularity strength of the total graph of star graph $K_{1,n}$ ($n \geq 2$) :

Theorem 2.1: for $n \geq 2$

$$tes(T(K_{1,n})) \leq \begin{cases} 2n, & for \quad n = 2,4 \\ 2n-1, & for \quad n = 3 \\ 2n+1, & for \quad n = 5 \\ \frac{n(n-1)}{2}+1, & for \quad n \text{ is even}, \; n \neq 2,4 \\ \frac{n(n-1)}{2}, & for \quad n \text{ is odd}, \; n \neq 3,5 \end{cases}$$

Proof: Let $G \cong T(K_{1,n})$, $n \geq 2$, be defined on

$V = \{u, u_i, v_i : 1 \leq i \leq n\}$

and $E = \{ uu_i , uv_i , u_iv_i , v_iv_{i+k} : 1 \leq i \leq n, k = 1, 2,..., \left\lceil \frac{n-1}{2} \right\rceil \}$

where i is taken modulo n (replacing 0 by n) and $K_{1,n}$ is the star graph.

Here G has 2n+1 vertices and $\frac{n(n+5)}{2}$ edges.

A function f is defined on V∪E as follows:

for $1 \le i \le n$,   f (u) = 1

$$f(u_i) = \left\lceil \frac{n}{2} \right\rceil$$

$$f(v_i) = \begin{cases} 2n, & \text{if } n \text{ is even} \\ 2n-1, & \text{if } n = 3 \\ 2n+1, & \text{if } n \text{ is odd}, n \neq 3 \end{cases}$$

and $$f(uu_i) = \begin{cases} \frac{n}{2} + i, & \text{if } n \text{ is even} \\ i, & \text{if } n = 3 \\ \left\lceil \frac{n}{2} \right\rceil + i, & \text{if } n \text{ is odd}, n \neq 3 \end{cases}$$

$$f(uv_i) = i$$

$$f(u_iv_i) = \begin{cases} \frac{n}{2} + i + 1, & \text{if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil + i, & \text{if } n \text{ is odd} \end{cases}$$

and $$f(v_iv_{i+k}) = \begin{cases} n(k-1) + i + 1, & \text{if } n \text{ is even} \\ i + 2, & \text{if } n = 3 \\ n(k-1) + i, & \text{if } n \text{ is odd}, n \neq 3 \end{cases}$$

where $k = 1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil$.

Since wt $(uv_i) = f(u) + f(uu_i) + f(u_i)$

$$= \begin{cases} n + 1 + i, & \text{if } n \text{ is even} \\ \left\lceil \frac{n}{2} \right\rceil + 1 + i, & \text{if } n = 3 \\ 2\left\lceil \frac{n}{2} \right\rceil + 1 + i, & \text{if } n \text{ is odd}, n \neq 3 \end{cases}$$

wt $(uv_i) = f(u) + f(uv_i) + f(v_i)$

$$= \begin{cases} 2n + 1 + i, & \text{if } n \text{ is even} \\ 2n + i, & \text{if } n \text{ is odd} \\ 2n + 2 + i, & \text{if } n \text{ is odd}, n \neq 3 \end{cases}$$

wt $(u_iv_i) = f((u_i) + f(u_iv_i) + f(v_i)$

$$= \begin{cases} 3n - 1 + i, & \text{if } n \text{ is even} \\ 2n - 1 + 2\lceil \frac{n}{2} \rceil + i, & \text{if } n = 3 \\ 2n + 1 + 2\lceil \frac{n}{2} \rceil + i, & \text{if } n \text{ is odd}, n \neq 3 \end{cases}$$

$wt(v_i v_{i+k}) = f(v_i) + f(v_i v_{i+k}) + f(v_{i+k})$

$$= \begin{cases} n(k+3) + i + 1, & \text{if } n \text{ is even} \\ 4n + i, & \text{if } n = 3 \\ n(k+3) + i + 2, & \text{if } n \text{ is odd}, n \neq 3 \end{cases}$$

where i + k subscripts modulo n,

The weights of the edges of $T(K_{1,n})$ under the labeling f are distinct and f is a mapping from

$V \cup E$ into $\{1,2,\ldots,2n\}$, if n = 2 and 4

$\{1,2,\ldots,2n-1\}$, if n = 3

$\{1,2,\ldots,2n+1\}$, if n = 5

$\{1,2,\ldots,\frac{n(n-1)}{2}+1\}$, if n is even, $n \neq 2,4$

and $\{1, 2,\ldots, \frac{n(n-1)}{2}\}$, if n is odd, $n \neq 3,5$.

Therefore total labeling of f has the required properties of all edge irregular total labeling.

Hence, we have

$$tes(T(S_n)) \leq \begin{cases} 2n, & \text{for } n = 2,4 \\ 2n - 1, & \text{for } n = 3 \\ 2n + 1, & \text{for } n = 5 \\ \frac{n(n-1)}{2} + 1, & \text{for } n \text{ is even}, n \neq 2,4 \\ \frac{n(n-1)}{2}, & \text{for } n \text{ is odd}, n \neq 3,5 \end{cases}$$

Theorem 2.2: $tvs(T(K_{1,n})) \leq n, n \geq 2$.

Proof: Let $G \cong T(K_{1,n})$, for $n \geq 2$ be defined as in Theorem 2.1.

Now consider a function f defined on $V \cup E$ as follows:

for $1 \leq i \leq n$, $f(u) = 1$

$f(u_i) = n-1$, $n \geq 2$

$f(v_i) = 2$

and $f(uu_i) = 1$

$f(uv_i) = 1$

$f(u_iv_i) = i$

$f(v_iv_{i+k}) = 2.$

where $k = 1, 2, \ldots, \left\lceil \frac{n-1}{2} \right\rceil$ and i is taken modulo n

Since $wt(u) = f(u) + \sum_{i=1}^{n} f(uu_i) + \sum_{i=1}^{n} f(uv_i)$

$= 1 + n + n$

$= 2n + 1$

$wt(u_i) = f(u_i) + f(uu_i) + f(u_iv_i)$

$= n + i$

and $wt(v_i) = f(v_i) + f(uv_i) + f(u_iv_i) + \sum f(v_iv_{i+k})$

$= 3 + i + \sum_{k=1}^{n-1} 2$

$= 3 + i + 2(n-1)$

$= 2n + i + 1$

The weight of the vertices of $T(K_{1,n})$ under the labeling f are distinct and f is a mapping from $V \cup E$ into $\{1, 2, \ldots, n\}$. Clearly the total labeling f has required properties of a vertex irregular total labeling.

Therefore $tvs(T(K_{1,n})) \leq n, \quad n \geq 2.$

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