

## Results on Coupled Fixed Points in $dqb$ -Metric Spaces

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**Abstract:** This paper presents the existence and uniqueness of the coupled fixed point results for mapping satisfying different contractive conditions in dislocated quasi  $b$ -metric spaces.

### Introduction and Preliminaries

**Definition 1.** [4] Let  $X$  be a non-empty set. Let the mapping  $d: X \times X \rightarrow [0, \infty)$  and constant  $k \geq 1$  satisfy following conditions:

1.  $d(x, y) = 0 = d(y, x) \Rightarrow x = y, \forall x, y \in X$
2.  $d(x, y) \leq k[d(x, z) + d(z, y)], \forall x, y, z \in X.$

Then  $d$  is called a dislocated quasi- $b$ -metric and the ordered pair  $(X, d)$  is called dislocated quasi- $b$ -metric space or in short  $dqb$ -metric space. The constant  $k$  is called coefficient of  $(X, d)$ .

It is clear that  $b$ -metric spaces, quasi- $b$ -metric spaces and  $b$ -metric-like spaces are  $dqb$ -metric spaces but converse is not true.

**Example 2.** [7] Let  $X = \mathbb{R}^+$  and for  $p > 1, d: X \times X \rightarrow [0, \infty)$  be defined as,

$$d(x, y) = |x - y|^p + |x|^p, \quad \forall x, y \in X.$$

Then  $(X, d)$  is  $dqb$ -metric space with  $k = 2^p > 1$ . It is easy to check that  $(X, d)$  is not  $b$ -metric space and also not dislocated quasi-metric space.

**Example 3.** [4] Let  $X = \mathbb{R}$  and suppose,

$$d(x, y) = |2x - y|^2 + |2x + y|^2, \quad \forall x, y \in \mathbb{R}.$$

Then  $(X, d)$  is  $dqb$ -metric space with coefficient  $k = 2$ . Clearly,  $(X, d)$  is not a quasi- $b$ -metric space. Also  $(X, d)$  is not dislocated quasi-metric space.

**Definition 4.** [4] A sequence  $\{x_n\}$  in a  $dqb$ -metric space  $(X, d)$ ,  $dqb$ -converges to  $x \in X$  if

$$\lim_{n \rightarrow \infty} d(x_n, x) = 0 = \lim_{n \rightarrow \infty} d(x, x_n).$$

In this case  $x$  is called  $dqb$ -limit of  $\{x_n\}$  and  $\{x_n\}$  is said to be  $dqb$ -convergent to  $x$ , written as  $x_n \rightarrow x$ .

**Definition 5.** A sequence  $\{x_n\}$  in a  $dqb$ -metric space  $(X, d)$  is called as  $dqb$ -Cauchy sequence if

$$\lim_{n,m \rightarrow \infty} d(x_n, x_m) = 0 = \lim_{n,m \rightarrow \infty} d(x_m, x_n).$$

**Definition 6.** [4] A  $dqb$ -metric space  $(X, d)$  is said to be  $dqb$ -complete if every  $dqb$ -Cauchy sequence in it is  $dqb$ -convergent in  $X$ .

**Definition 7.** [3] A subset  $M$  of  $X$  is said to be closed if and only if for each sequence  $\{x_n\}$  in  $X$  which converges to an element  $x \in M$ .

**Definition 8.** [3] A subset  $S$  of  $X$  is bounded if there exists  $M \in (0, \infty)$  such that  $d(x, y) \leq M$  for all  $x, y \in S$ .

**Proposition 9.** [9] Every subsequence of a  $dqb$ -convergent sequence in a  $dqb$ -metric space  $(X, d)$  is  $dqb$ -convergent sequence.

**Proposition 10.** [9] Every subsequence of a  $dqb$ -Cauchy sequence in a  $dqb$ -metric space  $(X, d)$  is  $dqb$ -Cauchy sequence.

**Proposition 11.** [9] If  $(X, d)$  is a  $dqb$ -metric space, then a function  $f: X \rightarrow X$  is continuous if and only if  $x_n \rightarrow x \Rightarrow f x_n \rightarrow f x$ .

**Lemma 12.** [9] Limit of a  $dqb$ -convergent sequence in  $dqb$ -metric space is unique.

**Proposition 13.** [9] If  $u$  is limit of some  $dqb$ -convergent sequence in a  $dqb$ -metric space  $(X, d)$  then  $d(u, u) = 0$ .

**Definition 14.** [2] An element  $(x, y) \in X \times X$  is called coupled fixed point of the mapping  $T: X \times X \rightarrow X$  if  $T(x, y) = x$  and  $T(y, x) = y$ .

**Definition 15.** An element  $(x, y) \in X \times X$  is called coupled coincidence point of the mapping  $T: X \times X \rightarrow X$  and  $h: X \rightarrow X$  if  $T(x, y) = hx$  and  $T(y, x) = hy$ .

**Definition 16.** Suppose  $X$  is a non-empty set. Then the mappings  $T: X \times X \rightarrow X$  and  $h: X \rightarrow X$  are commutative if  $hT(x, y) = T(hx, hy)$  for all  $x, y \in X$ .

**Lemma 17.** Let  $(X, d)$  be a  $dqb$ -metric space. Let  $F: X \times X \rightarrow X$  and  $g: X \rightarrow X$  be two mappings such that

$$d(F(x, y), F(u, v)) \leq \alpha[d(gx, gu) + d(gy, gv)],$$

for all  $x, y, u, v \in X$ . Assume that  $(x, y)$  is a coupled coincidence point of the mappings  $F$  and  $g$ . If  $2\alpha k < 1$ , with  $\alpha \in [0, 1)$  and  $k \geq 1$ . Then  $F(x, y) = gx = gy = F(y, x)$ .

## Main Results

**Theorem 18.** Let  $(X, d)$  be a  $dqb$ -metric space. Let  $T: X \times X \rightarrow X$  and  $f: X \rightarrow X$  be two continuous mappings such that

$$d(T(x, y), T(u, v)) \leq \alpha d(fx, fu) + \beta d(fy, fv), \quad (2.1) \text{ for all } x, y, u, v \in X, \text{ where } \alpha \text{ and } \beta$$

are constants such that  $0 < \alpha + \beta < \frac{1}{k}$ . Assume that  $T$  and  $f$  satisfy the following conditions:

1.  $f(X)$  is a complete  $dqb$ -metric space.

2.  $f$  commutes with  $T$ .
3.  $T(X \times X) \subseteq g(X)$ .

Then there exists unique coupled fixed point of  $T$  in  $X$ .

*Proof.* Let  $(x_0, y_0)$  be any arbitrarily fixed point in  $X \times X$ . Using (iii), we find  $(x_1, y_1) \in X \times X$  such that  $T(x_0, y_0) = fx_1$  and  $T(y_0, x_0) = fy_1$ . If we continue this way, we get two sequences of points in  $X$  namely  $\{x_n\}$  and  $\{y_n\}$ , with  $fx_{n+1} = T(x_n, y_n)$  and  $fy_{n+1} = T(y_n, x_n)$ . Using inequality (2.1), for arbitrary  $n \in N$ , we have

$$\begin{aligned}d(fx_n, fx_{n+1}) &= d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\leq \alpha d(fx_{n-1}, fx_n) + \beta d(fy_{n-1}, fy_n).\end{aligned}$$

Similarly,

$$\begin{aligned}d(fy_n, fy_{n+1}) &= d(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\ &\leq \alpha d(fy_{n-1}, fy_n) + \beta d(fx_{n-1}, fx_n).\end{aligned}$$

Now summing up these two inequalities, we get

$$d(fx_n, fx_{n+1}) + d(fy_n, fy_{n+1}) \leq (\alpha + \beta)[d(fx_{n-1}, fx_n) + d(fy_{n-1}, fy_n)].$$

Thus writing,  $d_n = d(fx_n, fx_{n+1}) + d(fy_n, fy_{n+1})$ , we get

$$d_n \leq (\alpha + \beta)d_{n-1}.$$

Let  $\alpha + \beta = \gamma$ . Then

$$0 \leq d_n \leq \gamma d_{n-1}, \quad \forall n \in N.$$

By induction on  $n$ , we get

$$0 \leq d_n \leq \gamma^n d_0, \quad \forall n \in N. \tag{2.2}$$

Now, for  $m, n \in N$  and  $n < m$ , using triangular inequality of the definition,

$$d(fx_n, fx_m) \leq kd(fx_n, fx_{n+1}) + k^2d(fx_{n+1}, fx_{n+2}) + \dots + k^{m-n}d(fx_{m-1}, fx_m).$$

Similarly,

$$d(fy_n, fy_m) \leq kd(fy_n, fy_{n+1}) + k^2d(fy_{n+1}, fy_{n+2}) + \dots + k^{m-n}d(fy_{m-1}, fy_m).$$

Adding these inequalities and using (2.2), we get

$$\begin{aligned}d(fx_n, fx_m) + d(fy_n, fy_m) &\leq k[d(fx_n, fx_{n+1}) + d(fy_n, fy_{n+1})] \\ &\quad + k^2[d(fx_{n+1}, fx_{n+2}) + d(fy_{n+1}, fy_{n+2})] \\ &\quad + \dots + k^{m-n}[d(fx_{m-1}, fx_m) + d(fy_{m-1}, fy_m)] \\ &\leq [k\gamma^n + k^2\gamma^{n+1} + \dots + k^{m-n}\gamma^{m-1}]d_0 \\ &= [(k\gamma)\gamma^{n-1} + (k\gamma)^2\gamma^{n-1} + \dots + (k\gamma)^{m-n}\gamma^{n-1}]d_0 \\ &\leq (m-n)\gamma^{n-1}d_0 \\ &\leq \gamma^{n-1}\eta, \quad \text{where } \eta \geq (m-n)d_0.\end{aligned}$$

Taking limit as  $n, m \rightarrow \infty$ , we get

$$\lim_{m,n \rightarrow \infty} d(fx_n, fx_m) = 0 \text{ and } \lim_{m,n \rightarrow \infty} d(fy_n, fy_m) = 0. \quad (2.3)$$

On the similar lines using equation (2.1), for arbitrary  $n \in N$ , we can write

$$\begin{aligned} d(fx_{n+1}, fx_n) &= d(T(x_n, y_n), T(x_{n-1}, y_{n-1})) \\ &\leq \alpha d(fx_n, fx_{n-1}) + \beta d(fy_n, fy_{n-1}). \end{aligned}$$

Similarly,

$$\begin{aligned} d(fy_{n+1}, fy_n) &= d(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \\ &\leq \alpha d(fy_n, fy_{n-1}) + \beta d(fx_n, fx_{n-1}). \end{aligned}$$

Now summing up these two inequalities, we get

$$d(fx_{n+1}, fx_n) + d(fy_{n+1}, fy_n) \leq (\alpha + \beta)[d(fx_n, fx_{n-1}) + d(fy_n, fy_{n-1})].$$

Thus writing,  $d'_n = d(fx_{n+1}, fx_n) + d(fy_{n+1}, fy_n)$ , we get

$$d'_n \leq (\alpha + \beta)d'_{n-1}.$$

Let  $\alpha + \beta = \gamma$ . Then

$$0 \leq d'_n \leq \gamma d'_{n-1}, \forall n \in N.$$

By induction on  $n$ , we get

$$0 \leq d'_n \leq \gamma^n d'_0, \forall n \in N. \quad (2.4)$$

Now, for  $m, n \in N$  and  $n < m$ , using triangular inequality of the definition,

$$d(fx_m, fx_n) \leq kd(fx_m, fx_{m-1}) + k^2 d(fx_{m-1}, fx_{m-2}) + \dots + k^{m-n} d(fx_{m-n+1}, fx_n)$$

Similarly,

$$d(fy_m, fy_n) \leq kd(fy_m, fy_{m-1}) + k^2 d(fy_{m-1}, fy_{m-2}) + \dots + k^{m-n} d(fy_{m-n+1}, fy_n).$$

Adding these inequalities and using equation (2.4)

$$\begin{aligned} d(fx_m, fx_n) + d(fy_m, fy_n) &\leq k[d(fx_{n+1}, fx_n) + d(fy_{n+1}, fy_n)] \\ &\quad + k^2[d(fx_{n+2}, fx_{n+1}) + d(fy_{n+2}, fy_{n+1})] + \dots \\ &\quad + k^{m-n}[d(fx_m, fx_{m-1}) + d(fy_m, fy_{m-1})] \\ &\leq [k\gamma^n + k^2\gamma^{n+1} + \dots + k^{m-n}\gamma^{m-1}]d'_0 \\ &= [(k\gamma)\gamma^{n-1} + (k\gamma)^2\gamma^{n-1} + \dots + (k\gamma)^{m-n}\gamma^{n-1}]d'_0 \\ &\leq (m-n)\gamma^{n-1}d'_0 \\ &\leq \gamma^{n-1}\eta, \text{ where } \eta \geq (m-n)d'_0. \end{aligned}$$

Taking limit as  $n, m \rightarrow \infty$ , we get

$$\lim_{m,n \rightarrow \infty} d(fx_m, fx_n) = 0 \text{ and } \lim_{m,n \rightarrow \infty} d(fy_m, fy_n) = 0. \quad (2.5)$$

Thus from equations (2.3) and (2.5),  $\{fx_n\}$  and  $\{fy_n\}$  are Cauchy sequences in  $f(X)$ . Since  $f(X)$  is complete  $dqb$ -metric space, there exist  $u, v \in f(X)$  such that  $fx_n \rightarrow u$  and  $fy_n \rightarrow v$ . Also  $T$  and  $f$  commute and are continuous. We claim that  $(u, v)$  is a coupled fixed point of  $T$ . As  $T$  and  $f$  commute,

$$ffx_{n+1} = fT(x_n, y_n) = T(fx_n, fy_n) \Rightarrow fu = T(u, v). \quad (2.6)$$

Similarly,

$$ff y_{n+1} = fT(y_n, x_n) = T(fy_n, fx_n) \Rightarrow fv = T(v, u). \quad (2.7)$$

Now, consider

$$\begin{aligned} d(T(u, v), fx_{n+1}) &= d(T(u, v), T(x_n, y_n)) \\ &\leq \alpha d(fu, fx_n) + \beta d(fv, fy_n). \end{aligned}$$

Also

$$\begin{aligned} d(T(v, u), fy_{n+1}) &= d(T(v, u), T(y_n, x_n)) \\ &\leq \alpha d(fv, fy_n) + \beta d(fu, fx_n). \end{aligned}$$

Adding these two inequalities and letting  $n \rightarrow \infty$ , we get

$$\begin{aligned} d(T(u, v), u) + d(T(v, u), v) &= \lim_{n \rightarrow \infty} [d(T(u, v), fx_{n+1}) + d(T(v, u), fy_{n+1})] \\ &\leq \alpha d(fu, u) + \beta d(fv, v). \end{aligned}$$

Using (2.6) and (2.7) we get

$$\begin{aligned} d(fu, u) + d(fv, v) &\leq \alpha d(fu, u) + \beta d(fv, v) \\ \Rightarrow d(fu, u) + d(fv, v) &< d(fu, u) + d(fv, v). \end{aligned}$$

since  $\alpha, \beta < 1$ . Which is a contradiction unless  $d(fu, u) = 0$  and  $d(fv, v) = 0$ . On the similar lines we can prove that  $d(u, fu) = 0$  and  $d(v, fv) = 0$ . Which leads us to conclude that  $fu = u$  and  $fv = v$ . This in turn gives  $T(u, v) = u$  and  $T(v, u) = v$ . Thus  $(u, v)$  is a coupled fixed point of  $T$  in  $X$ .

Now, we prove uniqueness of coupled fixed point  $(u, v)$ . Let, if possible,  $(u_1, v_1)$  be another coupled fixed point of  $T$  in  $X$ .

$$\begin{aligned} d(u, u_1) = d(T(u, v), T(u, v)) &\leq \alpha d(fu, fu) + \beta d(fv, fv) \\ &\leq \alpha d(u, u) + \beta d(v, v) = 0. \end{aligned}$$

Also,

$$\begin{aligned} d(u_1, u) = d(T(u, v), T(u, v)) &\leq \alpha d(fu, fu) + \beta d(fv, fv) \\ &\leq \alpha d(u, u) + \beta d(v, v) = 0. \end{aligned}$$

Thus  $u = u_1$ . Also, consider

$$d(v, v_1) = d(T(v, u), T(v, u)) \leq \alpha d(fv, fv) + \beta d(fu, fu) \leq \alpha d(v, v) + \beta d(u, u) = 0.$$

$$d(v_1, v) = d(T(v, u), T(v, u)) \leq \alpha d(fv, fv) + \beta d(fu, fu) \leq \alpha d(v, v) + \beta d(u, u) = 0.$$

Thus  $v = v_1$ . Hence  $(u, v) = (u_1, v_1)$ . And thus uniqueness of coupled fixed point of  $T$  in  $X$  is established.  $\square$

Define  $\Phi$  and  $\Psi$  as follows

$$\Phi = \{\phi: [0, \infty) \rightarrow [0, \infty) \mid \phi \text{ is continuous and non-decreasing, } \phi(t) = 0 \Leftrightarrow t = 0, \\ \phi(t + s) \leq \phi(t) + \phi(s), \forall s, t \in [0, \infty)\}.$$

$$\Psi = \left\{ \psi: [0, \infty) \rightarrow [0, \infty) \mid \lim_{t \rightarrow r} \psi(t) > 0, \forall r > 0 \text{ and } \lim_{t \rightarrow 0^+} \psi(t) = 0 \right\}.$$

**Theorem 19.** Let  $(X, d)$  be a complete  $dqb$ -metric space. Let  $T : X \times X \rightarrow X$  be continuous mappings such that

$$\phi \left( d(T(x, y), T(u, v)) \right) \leq \frac{1}{2} \phi(d(x, u) + d(y, v)) - \psi \left( \frac{d(x, u) + d(y, v)}{2} \right) \quad (2.8)$$

for all  $x, y, u, v \in X$ . Then  $T$  has a coupled fixed point in  $X$ .

*Proof.* Let  $(x_0, y_0)$  be any arbitrarily fixed point in  $X \times X$ . We can find  $(x_1, y_1) \in X \times X$  such that  $T(x_0, y_0) = x_1$  and  $T(y_0, x_0) = y_1$ . If we continue this way, we get two sequences of points in  $X$  namely  $\{x_n\}$  and  $\{y_n\}$ , with  $x_{n+1} = T(x_n, y_n)$  and  $y_{n+1} = T(y_n, x_n)$ . We use  $p_n = d(x_n, x_{n+1})$ ,  $q_n = d(y_n, y_{n+1})$ ,  $p'_n = d(x_{n+1}, x_n)$ ,  $q'_n = d(y_{n+1}, y_n)$ . Also  $d_n = p_n + q_n$ ,  $d'_n = p'_n + q'_n$ . Using inequality (2.8), for arbitrary  $n \in \mathbb{N}$ , we have

$$\begin{aligned} \phi(p_n) = \phi(d(x_n, x_{n+1})) &= d(T(x_{n-1}, y_{n-1}), T(x_n, y_n)) \\ &\leq \frac{1}{2} \phi(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) - \psi(d(x_{n-1}, x_n) + d(y_{n-1}, y_n)) \\ &= \frac{1}{2} \phi(p_{n-1} + q_{n-1}) - \psi \left( \frac{p_{n-1} + q_{n-1}}{2} \right) \end{aligned} \quad (2.9)$$

Similarly

$$\begin{aligned} \phi(q_n) = \phi(d(y_n, y_{n+1})) &= d(T(y_{n-1}, x_{n-1}), T(y_n, x_n)) \\ &\leq \frac{1}{2} \phi(d(y_{n-1}, y_n) + d(x_{n-1}, x_n)) - \psi(d(y_{n-1}, y_n) + d(x_{n-1}, x_n)) \\ &= \frac{1}{2} \phi(p_{n-1} + q_{n-1}) - \psi \left( \frac{p_{n-1} + q_{n-1}}{2} \right). \end{aligned} \quad (2.10)$$

Adding (2.9) and (2.10) and using property of  $\phi$ , we get

$$\begin{aligned} \phi(p_n + q_n) &\leq \phi(p_n) + \phi(q_n) \\ &= \frac{1}{2} \phi(p_{n-1} + q_{n-1}) - \psi \left( \frac{p_{n-1} + q_{n-1}}{2} \right) + \frac{1}{2} \phi(p_{n-1} + q_{n-1}) - \psi \left( \frac{p_{n-1} + q_{n-1}}{2} \right) \\ &\leq \phi(p_{n-1} + q_{n-1}) \\ \Rightarrow p_n + q_n &\leq p_{n-1} + q_{n-1}. \end{aligned} \quad (2.11)$$

Let us write  $d_n = p_n + q_n$ , and note that  $d_n$  is a non-increasing sequence of real numbers which is bounded below by 0 and hence there is some  $\gamma \geq 0$  such that

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} [p_n + q_n] = \gamma.$$

We claim that  $\gamma = 0$ . Suppose that  $\gamma > 0$ . Then taking the limit as  $n \rightarrow \infty$  of both and using properties of  $\phi$  and  $\psi$ , we have

$$\phi(\gamma) = \lim_{n \rightarrow \infty} \phi(d_n) \leq \lim_{n \rightarrow \infty} \left[ \phi(d_{n-1}) - 2\psi \left( \frac{d_{n-1}}{2} \right) \right] = \phi(\gamma) - 2 \lim_{d_{n-1} \rightarrow \gamma} \psi \left( \frac{d_{n-1}}{2} \right) < \phi(\gamma)$$

which is clearly a contradiction. Thus  $\gamma = 0$ , that is,

$$\lim_{n \rightarrow \infty} d_n = \lim_{n \rightarrow \infty} [p_n + q_n] = 0. \quad (2.12)$$

On the similar line, consider

$$\begin{aligned} \phi(p'_n) = \phi(d(x_{n+1}, x_n)) &= d(T(x_n, y_n), T(x_{n-1}, y_{n-1})) \\ &\leq \frac{1}{2} \phi(d(x_n, x_{n-1}) + d(y_n, y_{n-1})) - \psi(d(x_n, x_{n-1}) + d(y_n, y_{n-1})) \\ &= \frac{1}{2} \phi(p'_{n-1} + q'_{n-1}) - \psi\left(\frac{p'_{n-1} + q'_{n-1}}{2}\right). \end{aligned} \quad (2.13)$$

Similarly

$$\begin{aligned} \phi(q'_n) = \phi(d(y_{n+1}, y_n)) &= d(T(y_n, x_n), T(y_{n-1}, x_{n-1})) \\ &\leq \frac{1}{2} \phi(d(y_n, y_{n-1}) + d(x_n, x_{n-1})) - \psi(d(y_n, y_{n-1}) + d(x_n, x_{n-1})) \\ &= \frac{1}{2} \phi(p'_{n-1} + q'_{n-1}) - \psi\left(\frac{p'_{n-1} + q'_{n-1}}{2}\right). \end{aligned} \quad (2.14)$$

Adding (2.13) and (2.14) and using property of  $\phi$ , we get

$$\begin{aligned} \phi(p'_n + q'_n) &\leq \phi(p'_n) + \phi(q'_n) \\ &= \frac{1}{2} \phi(p'_{n-1} + q'_{n-1}) - \psi\left(\frac{p'_{n-1} + q'_{n-1}}{2}\right) + \frac{1}{2} \phi(p'_{n-1} + q'_{n-1}) - \psi\left(\frac{p'_{n-1} + q'_{n-1}}{2}\right) \\ &\leq \phi(p_{n-1} + q_{n-1}) \\ \Rightarrow p'_n + q'_n &\leq p'_{n-1} + q'_{n-1}. \end{aligned} \quad (2.15)$$

Let us write  $d'_n = p_n + q_n$ , and note that  $d'_n$  is a non-increasing sequence of real numbers which is bounded below by 0 and hence there is some  $\gamma' \geq 0$  such that

$$\lim_{n \rightarrow \infty} d'_n = \lim_{n \rightarrow \infty} [p'_n + q'_n] = \gamma'.$$

We claim that  $\gamma' = 0$ . Suppose that  $\gamma' > 0$ . Then taking the limit as  $n \rightarrow \infty$  of both and using properties of  $\phi$  and  $\psi$ , we have

$$\begin{aligned} \phi(\gamma) = \lim_{n \rightarrow \infty} \phi(d'_n) &\leq \lim_{n \rightarrow \infty} \left[ \phi(d'_{n-1}) - 2\psi\left(\frac{d'_{n-1}}{2}\right) \right] = \phi(\gamma') - 2 \lim_{d'_{n-1} \rightarrow \gamma'} \psi\left(\frac{d'_{n-1}}{2}\right) \\ &< \phi(\gamma'). \end{aligned}$$

which is clearly a contradiction. Thus  $\gamma' = 0$ , that is,

$$\lim_{n \rightarrow \infty} d'_n = \lim_{n \rightarrow \infty} [p'_n + q'_n] = 0. \quad (2.16)$$

Now, we wish to show that  $\{x_n\}$  and  $\{y_n\}$  are Cauchy sequences in  $X$ . Suppose to the contrary, that at least of  $\{x_n\}$  or  $\{y_n\}$  is not Cauchy sequence. Then there exists an  $\epsilon > 0$  for which we can find subsequences  $\{x_{n(i)}\}$ ,  $\{x_{m(i)}\}$  of  $\{x_n\}$  and  $\{y_{n(i)}\}$ ,  $\{y_{m(i)}\}$  of  $\{y_n\}$  with  $n(i) > m(i) \geq i$  such that

$$d(x_{n(i)}, x_{m(i)}) + d(y_{n(i)}, y_{m(i)}) \geq \epsilon.$$

For  $m(i)$ , we can choose  $n(i)$  in such a way that it is the smallest integer with  $n(i) > m(i) \geq i$  and satisfying above condition. Then

$$d(x_{n(i)-1}, x_{m(i)}) + d(y_{n(i)-1}, y_{m(i)}) < \epsilon. \quad (2.17)$$

Using above two inequalities and the triangle inequality, we have

$$\begin{aligned} \epsilon &\leq d(x_{n(i)}, x_{m(i)}) + d(y_{n(i)}, y_{m(i)}) \\ &\leq k[d(x_{n(i)}, x_{n(i)-1}) + d(x_{n(i)-1}, x_{m(i)}) + d(y_{n(i)}, y_{n(i)-1}) + d(y_{n(i)-1}, y_{m(i)})] \\ &\leq k[p'_{n(i)} + q'_{n(i)}] + k\epsilon. \end{aligned}$$

We write  $\alpha_i = d(x_{n(i)}, x_{m(i)}) + d(y_{n(i)}, y_{m(i)})$ . Now letting  $i \rightarrow \infty$  and using (2.12) and (2.16), we have

$$\lim_{i \rightarrow \infty} \alpha_i = \lim_{i \rightarrow \infty} [d(x_{n(i)}, x_{m(i)}) + d(y_{n(i)}, y_{m(i)})] = k\epsilon.$$

By the triangle inequality

$$\begin{aligned} \alpha_i &= d(x_{n(i)}, x_{m(i)}) + d(y_{n(i)}, y_{m(i)}) \\ &\leq k[d(x_{n(i)}, x_{n(i)+1}) + d(x_{n(i)+1}, x_{m(i)+1}) + d(x_{m(i)+1}, x_{m(i)}) \\ &\quad + d(y_{n(i)}, y_{n(i)+1}) + d(y_{n(i)+1}, y_{m(i)+1}) + d(y_{m(i)+1}, y_{m(i)})] \\ &= k[d_{n(i)} + d_{m(i)} + d(x_{n(i)+1}, x_{m(i)+1}) + d(y_{n(i)+1}, y_{m(i)+1})]. \end{aligned}$$

Using the property of  $\phi$ , we have

$$\begin{aligned} \phi(r_i) &= \phi\left(k\left(d_{n(i)} + d_{m(i)} + d(x_{n(i)+1}, x_{m(i)+1}) + d(y_{n(i)+1}, y_{m(i)+1})\right)\right) \\ &\leq \phi\left(k(d_{n(i)} + d_{m(i)})\right) + \phi\left(k\left(d(x_{n(i)+1}, x_{m(i)+1})\right)\right) + \phi\left(k\left(d(y_{n(i)+1}, y_{m(i)+1})\right)\right). \end{aligned}$$

Now, consider

$$\begin{aligned} \phi\left(d(x_{n(i)+1}, x_{m(i)+1})\right) &= \phi\left(k\left(d(T(x_{n(i)}, y_{n(i)}), T(x_{m(i)}, y_{m(i)}))\right)\right) \\ &\leq k\left[\frac{1}{2}\phi\left(d(x_{n(i)}, x_{m(i)}) + d(y_{n(i)}, y_{m(i)})\right) - \psi\left(\frac{d(x_{n(i)}, x_{m(i)}) + d(y_{n(i)}, y_{m(i)})}{2}\right)\right] \\ &= k\left[\frac{1}{2}\phi(\alpha_i) - \psi\left(\frac{\alpha_i}{2}\right)\right]. \end{aligned}$$

Similarly, we also have

$$\begin{aligned} \phi\left(d(y_{m(i)+1}, y_{n(i)+1})\right) &= \phi\left(k\left(d(T(y_{m(i)}, x_{m(i)}), T(y_{n(i)}, x_{n(i)}))\right)\right) \\ &\leq k\left[\frac{1}{2}\phi\left(d(y_{m(i)}, y_{n(i)}) + d(x_{m(i)}, x_{n(i)})\right) - \psi\left(\frac{d(y_{m(i)}, y_{n(i)}) + d(x_{m(i)}, x_{n(i)})}{2}\right)\right] \\ &= k\left[\frac{1}{2}\phi(\alpha_i) - \psi\left(\frac{\alpha_i}{2}\right)\right]. \end{aligned}$$

From above, we have

$$\phi(\alpha_i) \leq \phi\left(k(d_{n(i)} + d_{m(i)})\right) + k\phi(\alpha_i) - 2k\psi\left(\frac{\alpha_i}{2}\right).$$

Letting  $i \rightarrow \infty$ , we have

$$\phi(k\epsilon) \leq k\phi(\epsilon) \leq \phi(0) + k\phi(\epsilon) - 2k \lim_{i \rightarrow \infty} \psi\left(\frac{\alpha_i}{2}\right) = k\phi(\epsilon) - 2k \lim_{\alpha_i \rightarrow \epsilon} \psi\left(\frac{\alpha_i}{2}\right) < k\phi(\epsilon)$$

a contradiction. This shows that  $\{x_n\}$  and  $\{y_n\}$  are Cauchy sequences. Since  $X$  is a complete metric space, there exist  $u, v \in X$  such that  $\lim_{n \rightarrow \infty} x_n = u$  and  $\lim_{n \rightarrow \infty} y_n = v$ . Taking the limit as  $n \rightarrow \infty$ , we get

$$u = \lim_{n \rightarrow \infty} x_n = \lim_{n \rightarrow \infty} T(x_{n-1}, y_{n-1}) = T\left(\lim_{n \rightarrow \infty} x_{n-1}, \lim_{n \rightarrow \infty} y_{n-1}\right) = T(u, v)$$

and

$$v = \lim_{n \rightarrow \infty} y_n = \lim_{n \rightarrow \infty} T(y_{n-1}, x_{n-1}) = T\left(\lim_{n \rightarrow \infty} y_{n-1}, \lim_{n \rightarrow \infty} x_{n-1}\right) = T(v, u).$$

Therefore  $u = T(u, v)$  and  $v = T(v, u)$ . Thus we proved that  $T$  has a coupled fixed point in  $X$ .  $\square$

**Theorem 20.** Let  $(X, d)$  be a complete  $dqb$ -metric space with coefficient  $k$ . Let  $T : X \times X \rightarrow X$  be continuous mappings such that

$$d(T(x, y), T(u, v)) \leq \alpha d(x, T(x, y)) + \beta d(u, T(u, v)) \quad (2.18)$$

for all  $x, y, u, v \in X$  where  $0 < \alpha + \beta < \frac{1}{k}$ . Then  $T$  has unique coupled fixed point in  $X$ .

*Proof.* Let  $(x_0, y_0)$  be any arbitrarily fixed point in  $X \times X$ . We construct two sequences  $\{x_n\}$  and  $\{y_n\}$  such that  $x_n = T(x_{n-1}, y_{n-1})$  and  $y_n = T(y_{n-1}, x_{n-1})$ . Using equation [\(2.18\)](#), we can write

$$\begin{aligned} d(x_1, x_2) &= d(T(x_0, y_0), T(x_1, y_1)) \\ &\leq \alpha d(x_0, T(x_0, y_0)) + \beta d(x_1, T(x_1, y_1)) \\ &\leq \alpha d(x_0, x_1) + \beta d(x_1, x_2) \\ \Rightarrow d(x_1, x_2) &\leq \frac{\alpha}{1 - \beta} d(x_0, x_1). \end{aligned}$$

Put  $\frac{\alpha}{1 - \beta} = \gamma < 1$ . Then,

$$d(x_1, x_2) \leq \gamma d(x_0, x_1). \quad (2.19)$$

Now, consider

$$\begin{aligned} d(x_2, x_3) &= d(T(x_1, y_1), T(x_2, y_2)) \\ &\leq \alpha d(x_1, T(x_1, y_1)) + \beta d(x_2, T(x_2, y_2)) \\ &\leq \alpha d(x_1, x_2) + \beta d(x_2, x_3) \\ \Rightarrow d(x_2, x_3) &\leq \frac{\alpha}{1 - \beta} d(x_1, x_2). \end{aligned}$$

Using equation [\[7.31\]](#) this implies that

$$d(x_2, x_3) \leq \gamma^2 d(x_0, x_1).$$

Proceeding in this way, in general, we can write,

$$d(x_n, x_{n+1}) \leq \gamma^n d(x_0, x_1). \quad (2.20)$$

Similarly,

$$d(y_n, y_{n+1}) \leq \gamma^n d(y_0, y_1). \quad (2.21)$$

Now, for  $m, n \in N$  and  $n < m$ , using triangular inequality of the definition and using inequality [7.32],

$$\begin{aligned} d(x_n, x_m) &\leq kd(x_n, x_{n+1}) + k^2d(x_{n+1}, x_{n+2}) + \dots + k^{m-n}d(x_{m-1}, x_m) \\ &\leq [k\gamma^n + k^2\gamma^{n+1} + \dots + k^{m-n}\gamma^{m-1}]d(x_0, x_1) \\ &= [(k\gamma)\gamma^{n-1} + (k\gamma)^2\gamma^{n-1} + \dots + (k\gamma)^{m-n}\gamma^{n-1}]d(x_0, x_1) \\ &\leq (m-n)\gamma^{n-1}d(x_0, x_1) \\ &\leq \gamma^{n-1}\eta, \quad \text{where } \eta \geq (n-m)d(x_0, x_1). \end{aligned}$$

Taking limit as  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} d(x_n, x_m) = 0. \quad (2.22)$$

Similarly, consider

$$\begin{aligned} d(y_n, y_m) &\leq kd(y_n, y_{n+1}) + k^2d(y_{n+1}, y_{n+2}) + \dots + k^{m-n}d(y_{m-1}, y_m) \\ &\leq [k\gamma^n + k^2\gamma^{n+1} + \dots + k^{m-n}\gamma^{m-1}]d(y_0, y_1) \\ &= [(k\gamma)\gamma^{n-1} + (k\gamma)^2\gamma^{n-1} + \dots + (k\gamma)^{m-n}\gamma^{n-1}]d(y_0, y_1) \\ &\leq (m-n)\gamma^{n-1}d(y_0, y_1) \\ &\leq \gamma^{n-1}\eta, \quad \text{where } \eta \geq (m-n)d(y_0, y_1). \end{aligned}$$

Taking limit as  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} d(y_n, y_m) = 0. \quad (2.23)$$

On the similar line using inequality [2.18], consider

$$\begin{aligned} d(x_2, x_1) &= d(T(x_1, y_1), T(x_0, y_0)) \\ &\leq \alpha d(x_1, T(x_1, y_1)) + \beta d(x_0, T(x_0, y_0)) \\ &\leq \alpha d(x_1, x_2) + \beta d(x_0, x_1). \end{aligned}$$

Using equation [2.19], we get

$$d(x_2, x_1) \leq (\alpha\gamma + \beta)d(x_0, x_1).$$

Let us write  $\alpha\gamma + \beta = \delta < 1$ . Then

$$d(x_2, x_1) \leq \delta d(x_0, x_1).$$

Proceeding in this way, in general, we can write,

$$d(x_{n+1}, x_n) \leq \delta^n d(x_0, x_1). \quad (2.24)$$

Similarly,

$$d(y_{n+1}, y_n) \leq \delta^n d(y_0, y_1). \quad (2.25)$$

Now, for  $m, n \in N$  and  $n < m$ , using triangular inequality of the definition and using equation [7.35]

$$\begin{aligned} d(x_m, x_n) &\leq kd(x_{n+1}, x_n) + k^2d(x_{n+2}, x_{n+1}) + \dots + k^{m-n}d(x_m, x_{m-1}) \\ &\leq [k\delta^n + k^2\delta^{n+1} + \dots + k^{m-n}\delta^{m-1}]d(x_0, x_1) \\ &= [(k\delta)\delta^{n-1} + (k\delta)^2\delta^{n-1} + \dots + (k\delta)^{m-n}\delta^{n-1}]d(x_0, x_1) \\ &\leq (m-n)\delta^{n-1}d(x_0, x_1) \\ &\leq \delta^{n-1}\eta, \quad \text{where } \eta \geq (m-n)d(x_0, x_1). \end{aligned}$$

Taking limit as  $n, m \rightarrow \infty$ , we get

$$\lim_{m, n \rightarrow \infty} d(x_m, x_n) = 0. \quad (2.26)$$

Similarly, consider

$$\begin{aligned} d(y_m, y_n) &\leq kd(y_{n+1}, y_n) + k^2d(y_{n+2}, y_{n+1}) + \dots + k^{m-n}d(y_m, y_{m-1}) \\ &\leq [k\delta^n + k^2\delta^{n+1} + \dots + k^{m-n}\delta^{m-1}]d(y_0, y_1) \\ &= [(k\delta)\delta^{n-1} + (k\delta)^2\delta^{n-1} + \dots + (k\delta)^{m-n}\delta^{n-1}]d(y_0, y_1) \\ &\leq (m-n)\delta^{n-1}d(y_0, y_1) \\ &\leq \delta^{n-1}\eta, \quad \text{where } \eta \geq (m-n)d(y_0, y_1). \end{aligned}$$

Taking limit as  $n, m \rightarrow \infty$  we get

$$\lim_{m, n \rightarrow \infty} d(y_m, y_n) = 0.$$

Thus from equations (2.22), (2.23), (2.26) and (2.27),  $\{x_n\}$  and  $\{y_n\}$  are Cauchy sequences in  $X$ . Since  $X$  is complete  $dqb$ -metric space, there exist  $u, v \in X$  such that  $x_n \rightarrow u$  and  $y_n \rightarrow v$ . We claim that  $(u, v)$  is a coupled fixed point of  $T$ . As  $T$  is continuous,

$$x_{n+1} = T(x_n, y_n) \Rightarrow u = T(u, v). \quad (2.28)$$

Similarly,

$$y_{n+1} = T(y_n, x_n) \Rightarrow v = T(v, u). \quad (2.28)$$

Thus  $(u, v)$  is a coupled fixed point of  $T$  in  $X$ .

Now we prove uniqueness of coupled fixed point  $(u, v)$ . Let, if possible,  $(u_1, v_1)$  be another coupled fixed point of  $T$  in  $X$ .

$$\begin{aligned} d(u, u_1) = d(T(u, v), T(u, v)) &\leq \alpha d(u, T(u, v)) + \beta d(u, T(u, v)) \\ &\leq \alpha d(u, u) + \beta d(u, u) = 0. \end{aligned}$$

Also

$$\begin{aligned} d(u_1, u) = d(T(u, v), T(u, v)) &\leq \alpha d(u, T(u, v)) + \beta d(u, T(u, v)) \\ &\leq \alpha d(u, u) + \beta d(u, u) = 0. \end{aligned}$$

Thus  $u = u_1$ . Now, consider

$$\begin{aligned}d(v, v_1) = d(T(v, u), T(v, u)) &\leq \alpha d(v, T(v, u)) + \beta d(v, T(v, u)) \\ &\leq \alpha d(v, v) + \beta d(v, v) = 0.\end{aligned}$$

Also,

$$\begin{aligned}d(v_1, v) = d(T(v, u), T(v, u)) &\leq \alpha d(v, T(v, u)) + \beta d(v, T(v, u)) \\ &\leq \alpha d(v, v) + \beta d(v, v) = 0.\end{aligned}$$

Thus  $v = v_1$ . Hence  $(u, v) = (u_1, v_1)$ . And thus uniqueness of coupled fixed point of  $T$  in  $X$  is established.  $\square$

### References:

- [1] A. Beiranvand, S. Moradi, M. Omid and H. Pazandeh, Two Fixed-Point Theorems For Special Mappings, arXiv:0903.1504v1 [math.FA] 9 Mar 2009.
- [2] T. Bhaskar and V. Lakshmikantham, Fixed point theorem in partially ordered metric spaces and applications, *Nonlinear Anal.*, 65 (2006), 1379-1393.
- [3] A. Buayai, A. Wiwatwanich and A. Kaewkhao, Fixed Point For Cyclic Multi-Valued Mapping In Complete Dislocated Quasi-B-Metric Spaces, *International Journal of Pure and Applied*
- [4] C. Klin-eam and C. Suanoom, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic contractions, *Klin-eam and Suanoom Fixed Point Theory and Applications*, (2015)
- [5] S. Nadler, Jr., Multi-Valued Contraction Mappings, *Pacific Journal Of Mathematics*, Vol. 30, No. 2, 1969.
- [6] M. Rahman and M. Sarwar, Coupled Fixed Point Theorem in Dislocated Quasi b-Metric Spaces, *Commun. Nonlinear Anal. 2* (2016), 113-118.
- [7] M. Rahman and M. Sarwar, Dislocated quasi b-metric space and fixed point theorems, *Electronic Journal of Mathematical Analysis and Applications* Vol. 4(2) July 2016, pp. 16-24.
- [8] F. Sabetghadam, H. Masiha and A. Sanatpour, Some coupled fixed point theorems in cone metric spaces, *Fixed Point Theory and Applications*, Volume 2009, Article ID 125426, doi:
- [9] C. Suanoom, C. Klin-eam and S. Suantai, Dislocated quasi-b-metric spaces and fixed point theorems for cyclic weakly contractions, *J. Nonlinear Sci. Appl.* 9 (2016), 2779-2788.