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Study of Application of Adomian Decomposition Method to solve Heat Equation with Initial and Boundary Conditions

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Abstract:

This paper deals with the application of the decomposition method to solve the fractional homogeneous heat equation with initial and boundary value conditions. The fractional heat equation, governed by derivatives of non-integer order, serves as a vital model in describing anomalous diffusion processes and memory effects in various physical and biological systems. Adomian Decomposition Method (ADM) provides the solution which expressed as a rapidly converging series of functions without requiring discretization, linearization, or perturbation techniques. Several numerical examples are provided to validate the effectiveness and accuracy of the proposed approach.

Keyword: *Fractional homogeneous heat equation, Adomian Decomposition Method.*

Introduction:

The study of fractional differential equations has attracted mathematician and other researchers due its application in almost all the field such as in fluid mechanics, viscoelasticity, biology, physics, engineering and other areas of science. The primary reason for interest is that the exact description of most of the phenomena has been governed by equations involving fractional order derivatives. Physical phenomenon may depend not only the time instant but also the previous time history, which can be successfully modeled by use of the theory of derivatives and integrals of fractional order [1–5].

There have been several fundamental works on the fractional derivative and fractional differential equations, written by Oldham and Spanier [7], Miller and Ross [6], Poldubny [8] and others. Works of these researchers are an introduction to the theory of the fractional derivative and fractional differential equations which provide a systematic understanding of the fractional calculus such as the existence and the uniqueness, some analytical methods for solving fractional differential equations are introduced. There are many research papers on fractional differential equations which includes the development of an effective method for solving fractional differential equations. The work of Diethelm et al. is devoted to establishing numerical solutions of several classes of linear and nonlinear fractional differential equations [9-14].

Preliminaries and Notations:

Fractional Calculus:

In literature there are many definitions of fractional derivatives and integral [21-26] but the most frequently used are as given below.

Definition:

Caputo fractional derivative with order α for a function x(t) is defined as

$$^{c}D^{\alpha}_{t0}(x(t))=\frac{1}{\Gamma(m-\alpha)}\int_{t0}^{t}(t-\tau)^{m-\alpha-1}\,x^{m}(\tau)\,d\tau$$

where $0 \leq m - 1 \leq \alpha < m$, $m \in Z_{+}$, and t = t0 is the initial time and $\Gamma(.)$ is the Gamma function.

Definition :

Riemann-Liouville fractional integral of order $\alpha > 0$ for a f(x) :$(0,\infty)\rightarrow R$ is defined as

$$I^{\alpha}_{t0}\,(f(t)) = \frac{1}{\Gamma(\alpha)}\int_{t0}^{t}(t-\tau)^{\alpha-1} f(\tau)\,d\tau$$

Where $t = t_0$ is the initial time and $\Gamma(.)$ is the Gamma function.

Definition :

Riemann-Liouville fractional derivative with order α for a function x : R+ →R is defined as

$$^{RL}D^{\alpha}_{t0}(x(t))=\frac{1}{\Gamma(m-\alpha)}\frac{d^{m}}{dx^{m}}\int_{t0}^{t}(t-\tau)^{m-\alpha-1}\,x(\tau)\,d\tau$$

Where $0 \leq m - 1 \leq \alpha < m$, $m \in Z_{+}$, and t = t0 is the initial time and $\Gamma(.)$ is the Gamma function.

Definition :

The Mittag-Leffler function is defined as

$$E_{\alpha}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+1)}$$

Where $\alpha > 0$, $z \in C$. The two-parameter Mittag-Leffler function is defined as

$$E_{\alpha,\beta}(Z) = \sum_{k=0}^{\infty}\frac{z^{k}}{\Gamma(k\alpha+\beta)},\quad \alpha,\beta > 0,\ z \in C$$

There are some properties between fractional-order derivatives and fractional order integrals, which are expressed as follows.

(i) If $f \in C[0,\infty)$, then the Riemann–Liouville fractional order integral has the following important property:

$$I^{\alpha}(I^{\beta} f(t))= I^{\alpha+\beta} f(t)$$

where $\alpha > 0$ and $\beta > 0$.

(ii) For $\alpha > 0$, $t > 0$,

$$D^{\alpha}(I^{\alpha} f(t)) = f(t)$$

i.e. the Riemann–Liouville fractional derivative is the left inverse of the Riemann–Liouville fractional integral of the same order.

(iii)Let $\alpha > 0$, $n = [\alpha]+1$ and $f_{n-\alpha}(t) = (I^{n-\alpha}_{a} f)(t)$

Then fractional integrals and fractional derivatives have the following properties.

(1) If $f(t) \in L^{1}(a; b)$ and $f_{n-\alpha}(t) \in AC^{n}[a; b]$, then

$$(I^{\alpha}_{a}\,{}^{RL}D^{\alpha}_{a}\, f)(t) = f(t) - \sum_{j=1}^{n}\frac{f^{(n-j)}{n-\alpha(a)}}{\Gamma(a-j+1)}(t-a)^{\alpha-j},$$

holds almost everywhere in [a; b].

(2) If f(t∈ $AC^{n}[a; b]$, or $f(t) \in C^{n}[a; b]$, then

$$(I^{\alpha}_{a}\,{}^{c}D^{\alpha}_{a}\, f)(t) = f(t) - \sum_{k=0}^{n-1}\frac{f^{(k)}(a)}{k!}(t-a)^{k}.$$

(iv) Riemann–Liouville fractional derivative of a constant c is given by

$$D^{\alpha}(c) = \frac{ct^{-\alpha}}{\Gamma(1-\alpha)}$$

.One of the most important advantages of using a Caputo type fractional derivative is that the Caputo derivative of a constant is zero, which means this kind of derivative can be used to model the rate of change.

One Dimensional Heat Flow:

Many phenomena of physics, engineering, fluid mechanics, viscoelasticity and biology are expressed by partial differential equations PDEs. The PDE is termed a *Boundary Value Problem* (BVP) if the boundary conditions of the dependent variable u and some of its partial derivatives are often prescribed. However, the PDE is called an *Initial Value Problem* (IVP) if the initial conditions of the dependent variable u are prescribed at the starting time $t = 0$. Moreover, the PDE is termed *Initial-Boundary Value Problem* (IBVP) if both initial conditions and boundary conditions are prescribed.

The one dimensional heat flow will be focused on solving the PDE in conjunction with the prescribed initial and boundary conditions. The Adomian decomposition method [16–17] will be used to handle the heat flow PDEs.

Partial Differential Equation (PDE) that governs the heat flow in a rod. The PDE can be formally shown to satisfy

$$u_t = ku_{xx}, \qquad 0 < x < L,\ t > 0,$$

where $u \equiv u(x,t)$ represents the temperature of the rod at the position x at time t, and k is the thermal diffusivity of the material that measures the rod ability to heat conduction.

Adomian Decomposition Method:

The Adomian decomposition method was introduced and developed by George Adomian in [16-17]. A considerable amount of research work has been done recently in applying this method to a wide class of linear and nonlinear ordinary differential equations, partial differential equations and integral equations as well.

The formal steps of the decomposition method. A framework for implementing this method to solve the one dimensional heat equation.

Without loss of generality, we study the initial-boundary value problem

$$\begin{aligned} &\text{PDE} \quad ut = uxx,\ 0 < x < \pi,\ t > 0, \\ &\text{BC} \quad u(0,t) = 0,\ t \geq 0, \\ &\qquad\ \ u(L,t) = 0,\ t \geq 0, \\ &\text{IC} \quad u(x,0) = f(x),\ 0 \leq x \leq \pi, \end{aligned} \qquad \text{(A)}$$

to achieve our goal.

To begin our analysis, we first rewrite (A) in an operator form by

$L_t\, u(x,t) = L_x\, u(x,t)$,
where the differential operators L_t and L_x are defined by

$$L_t = \partial/\partial t\ ,\ L_x = \partial^2/\partial x^2\ .$$

(B)
It is obvious that the integral operators $L^{-1}{}_t$ and $L^{-1}{}_x$ exist and may be regarded as one and two-fold definite integrals respectively defined by

$$L^{-1}{}_t\,(\cdot) = \int_0^t (\cdot)dt\ ,\ L^{-1}{}_x\,(\cdot) = \iint_{00}^{xx} (.)dxdx$$

(C)
This means that

$$L^{-1}{}_t\, L_t\, u(x,t) = u(x,t) - u(x,0).$$

(D)
Applying $L^{-1}{}_t$ to both sides of (B) and using the initial condition we find

$$u(x,t) = f(x) + L^{-1}{}_t\,(L_x\, u(x,t)).$$

(E)
The decomposition method defines the unknown function $u(x,t)$ into a sum of components defined by the series

$$u(x,t) = \sum_{n=0}^{\infty} u_n(x,t),$$

(F)

One Dimensional Heat Flow:
where the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ are to be determined. Substituting (F) into both sides of (E) yields

$$\sum_{n=0}^{\infty} u_n(x,t) = f(x) + L^{-1}{}_t\,(L_x(\sum_{n=0}^{\infty} u(x,t)\ )),$$

(G)
or equivalently

$$u_0 + u_1 + u_2 + \cdots = f(x) + L^{-1}{}_t\,(L_x(u_0 + u_1 + u_2 + \cdots))$$

(H)
The decomposition method suggests that the zeroth component $u_0(x,t)$ is identified by the terms arising from the initial/boundary conditions and from source terms.
The remaining components of $u(x,t)$ are determined in a recursive manner such that each component is determined by using the previous component. Accordingly, we set the recurrence scheme

$$u_0(x,t) = f(x),$$

$u_{k+1}(x,t) = L^{-1}{}_t\,(L_x\,(u_k(x,t)))\ ,\ k \geq 0,$ (I)
for the complete determination of the components $un(x,t), n$ _ 0. In view of (G), the components $u_0(x,t), u_1(x,t), u_2(x,t), \cdots$ are determined individually by

$$u_0(x,t) = f(x),$$
$$u_1(x,t) = L^{-1}{}_t\, L_x(u_0) = f``(x)t,$$
$$u_2(x,t) = L^{-1}{}_t\, L_x(u_1) = f^{(4)}(x)\ t^2 / 2!\ ,$$
$$u^3(x,t) = L^{-1}t\ L_x(u_2) = f^{(6)}(x)t^3/3!,$$

… (J)
Other components can be determined in a like manner as far as we like. The accuracy level can be effectively improved by increasing the number of components determined. Having

determine -d the components $u_0, u_1, \cdots$, the solution $u(x,t)$ of the PDE is thus obtained in a series form given by

$$(x,t) = \sum_{n=0}^{\infty} f^{(2n)}(x)\, t^n/n!,$$

Obtained by substituting (I) into (D).[17,18]

Numerical Examples:

Example 1.

P.D.E $D_t^{\alpha} u(x,t) = u_{xx}(x,t),\ 0 < x < \pi, \quad t > 0$ **(2.1)**

B.C. $u(0,t) = 0, \quad t \geq 0,$

$u(\pi,t) = 0, \quad t \geq 0,$

I.C. $u(x,0) = \sin x,$

Where D_t^{α} denote Caputo derivative of order α.0 < α < 1.
Applying $D_t^{-\alpha} = J_t^{\alpha}$ on both side of the above equation (2.1).
$J_t^{\alpha}\,[D_t^{\alpha} u] = J_t^{\alpha}[\,u_{xx}\,]$

$u(x,t) = \sin x + J_t^{\alpha}[\,u_{xx}\,]$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = \sin x + J_t^{\alpha}[(\sum_{n=0}^{\infty} u_n)_{xx}]$$

Recursive relation is defined by

$$u_0 = \sin x$$
$$\sum_{k=0}^{\infty} u_{k+1} = J_t^{\alpha}[(\sum_{n=0}^{\infty} u_n)_{xx}],\ k \geq 0.$$
$$u_0 = \sin x$$
$$u_1 = J_t^{\alpha}[-\sin x]$$
$$u_1 = -\sin x \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$
$$u_2 = J_t^{\alpha}[-\sin x \frac{t^{\alpha}}{\Gamma(\alpha+1)}]$$
$$u_2 = \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$
$$u_3 = J_t^{\alpha}[-\sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}]$$
$$u_3 = -\sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$
$$\vdots$$
$$u(x,y) = \sin x - \sin x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \sin x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \sin x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots$$
$$= \sin x\,[1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots]$$

Exact solution is $\sin x\, e^{-t}$

Example 2:

P.D.E $D_t^{\alpha} u(x,t) = u_{xx}(x,t),\ 0 < x < \pi, \quad t > 0$ **(2.2)**

B.C. $u(0,t) = 4 + e^{-t}, \quad t \geq 0,$

$u(\pi, t) = 4 - e^{-t}, \quad t \geq 0,$

I.C. $\quad u(x, 0) = 4 + \cos x,$

Where D_t^{α} denote Caputo derivative of order α.0 < α < 1.
Applying $D_t^{-\alpha} = J_t^{\alpha}$ on both side of the above equation (2.2).

$$J_t^{\alpha}[D_t^{\alpha}u] = J_t^{\alpha}[u_{xx}]$$

$$u(x,t) = 4 + \cos x + J_t^{\alpha}[u_{xx}]$$

Adomian method defines the solution u by an infinite series of components given by

$$u = \sum_{n=0}^{\infty} u_n$$

Hence,

$$\sum_{n=0}^{\infty} u_n = 4 + \cos x + J_t^{\alpha}[(\sum_{n=0}^{\infty} u_n)_{xx}]$$

Recursive relation is defined by

$$u_0 = 4 + \cos x$$

$$\sum_{k=0}^{\infty} u_{k+1} = J_t^{\alpha}[(\sum_{n=0}^{\infty} u_n)_{xx}], \ k \geq 0.$$

$$u_0 = 4 + \cos x$$

$$u_1 = J_t^{\alpha}[-\cos x]$$

$$u_1 = -\cos x \frac{t^{\alpha}}{\Gamma(\alpha+1)}$$

$$u_2 = J_t^{\alpha}[-\cos x \frac{t^{\alpha}}{\Gamma(\alpha+1)}]$$

$$u_2 = \cos x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}$$

$$u_3 = J_t^{\alpha}[-\cos x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)}]$$

$$u_3 = -\cos x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)}$$

$$\vdots$$

$$u(x,y) = 4 + \cos x - \cos x \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \cos x \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \cos x \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \ldots$$

$$= 4 + \cos x [1 - \frac{t^{\alpha}}{\Gamma(\alpha+1)} + \frac{t^{2\alpha}}{\Gamma(2\alpha+1)} - \frac{t^{3\alpha}}{\Gamma(3\alpha+1)} + \cdots]$$

Exact solution is $4 + \cos x \, e^{-t}$.

Observation Table:

At	Example 1			Example 2		
	Exact Value	Approx Value	Absolute Error	Exact Value	Approx Value	Absolute Error
0.2	0.578930	0.455227	0.123703	4.578930	4.455227	0.123703
0.4	0.473988	0.391471	0.082516	4.473988	4.391471	0.082516
0.6	0.388068	0.352269	0.035799	4.388068	4.352269	0.035799
0.8	0.317724	0.324558	0.006834	4.317724	4.324558	0.006834

Conclusion:

Adomian Decomposition Method (ADM) produces the solution which is expressed as a rapidly converging series of functions without requiring discretization, linearization, or perturbation techniques. Approach of this method validates the effectiveness, accuracy and capability to handle the complexity of fractional heat equations.

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